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The Ring Division Self Duality

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Abstract

We present a simple construction of the instantonic type equation over octonions where its similarities and differences with the quaternionic case are very clear. We use the unified language of Clifford Algebra. We argue that our approach is the pure algebraic formulation of the geometric based soft Lie algebra. The topological criteria for the stability of our solution is given explicitly to establish its solitonic property. Many beautiful features of the parallelizable ring division spheres and Absolute Parallelism (AP) reveal their presence in our formulation.

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1 Introduction

During this century, physics has evolved a great deal. From general relativity to quantum fields and finally by Yang-Mills gauge fields, our mathematical tools have been amplified. It seems that our world has signed an agreement with Dirac's ideology "One can generalize his physics by generalizing his mathematics". For example, we know that our non-abelian gauge fields without any coupling to matter have a very rich topological structure. Instantons represent a remarkable bridge between modern mathematics and - eventually - physics. The extension of the 4 dimensional Euclidean Yang-Mills instanton [1] to higher dimensions was formulated by different groups [2, 3, 4, 5, 6]. Recently, such topics appeared to the surface with important application in string solitons [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18].

In this article, we would like to give a new construction for an octonionic instantonic type equation where the parallelism between such solutions in different dimensions is very transparent. Also, the topological stability is written explicitly to establish the solitonic, not just the integrability, property of such a model.

We use the following notations: $dim = 2^n$, roman letters (a, b, \dots) run from 1 to $(dim - 1)$ whereas Greek letters (α, β, \dots) run from 0 to $(dim - 1)$. We denote quaternions and octonions by Q and O respectively. The paper is organized as follows: in the next section, we give a brief review of some geometric properties of the ring division algebra then in section 3., we define a Clifford self duality which enable us to represent, in section 4., a novel construction of the eight dimensional instantons where we also compute its "torsionful cohomology group", unfortunately, our solution does not satisfy the YM equation of motion but

$$3\partial_\mu F_{\mu\nu} + [A_\mu, F_{\mu\nu}] = 0. \quad (1)$$

Then in section 5. we discuss the possibility of extending this formulation to higher dimensions. Lastly in section 6. we spell out the message of this paper.

2 The AP Structure

We have first to review some of the beautiful algebraic, geometric and topological properties of the ring division algebra. We will see that different branches of mathematics must be used together to give the correct physical picture. Using algebra alone and writing down a multiplication table is not at all sufficient.

One of the most important application of the ring division algebra in modern mathematics is its connection with the problem of finding the number of vector fields over any sphere. A problem that had been solved completely by Adams[19] and then the solution had been simplified and reformulated by different great mathematicians. Following Husemoller [20], one can simply say that the number of allowed vector fields $N-1$ that parallelize - i.e the needed number of linearly independent vector fields, that exist globally, and form a basis of - a sphere is related, on the one hand, to the ring division algebras which single out just three spheres S^1, S^3 and lastly S^7 . Whereas on the other hand, these $N-1$ vector fields are related to a $Cliff(0, N-1)$ structure expressed as $N \times N$ real

matrices. Again, which is only possible in 1,3 and 7 dimensions. Leading to $N=1,2,4,8$ the dimensions of the different ring division algebra. In [21, 22], We have shown how to construct a $\text{Cliff}(0,N-1)$ directly from the ring division algebra where they are simply given by

$$(E_i)_{\alpha\beta} = \delta_{i\alpha}\delta_{\beta 0} - \delta_{i\beta}\delta_{\alpha 0} + f_{i\alpha\beta}, \quad (2)$$

f_{ijk} is the ring division structure constant, then we have

$$\{E_i, E_j\} = -2\delta_{ij}. \quad (3)$$

Also note that

$$(E_i)^T = -E_i, \quad (E_i)^2 = -1. \quad (4)$$

At the level of quaternions, we have a complete algebraic isomorphism

$$E_i E_j = -\delta_{ij} + f_{ijk} E_k, \quad (5)$$

whereas for octonions, one finds

$$E_i E_j = -\delta_{ij} + f_{ijk} E_k - [E_i, 1|E_j], \quad (6)$$

we use Rotelli's notation[23] where right action is denoted by $1|E_i$.

But since the S^7 sphere is parallelized by octonions or by the above $\text{Cliff}(0,7)$ then they are geometrically isomorphic and this algebraic difference is just a reflection of comparing an associative algebra to a non-associative one. To be more convincing, let's review briefly this Absolute Parallelism (AP) structure i.e how our $\text{Cliff}(0,\text{dim}-1)$ parameterize the different ring division $S^{\text{dim}-1}$ spheres. AP spaces are non-trivial torsionfull manifolds. They have a very important characteristic: The vanishing of the parallelizable torsionfull connection i.e. the covariant derivative will be reduced to the standard derivative [24, 25]

$$D_\mu A_\nu = \partial_\mu A_\nu - \Gamma_{\mu\nu}^\alpha A_\alpha = \partial_\mu A_\nu. \quad (7)$$

We are going to demonstrate how this happens for the ring division spheres. **For quaternions or octonions, we can use either of the e_i or the E_i sets. The important thing is having a $\text{Cliff}(0,3)$ (4×4 matrices) and a $\text{Cliff}(0,7)$ (8×8 matrices) structure.**

Our line of attack is simply finding the parallelizable coordinates frame where the metric will have a flat basis. Working over R^{dim} , we have the following metric

$$ds^2 = dx_\mu dx^\mu, \quad (8)$$

embedding the $S^{\text{dim}-1}$ spheres amounts to impose the condition

$$x_\mu x^\mu = R^2, \quad (9)$$

which induces the Cartesian metric

$$ds^2 = \left(\delta_{mn} + \frac{y^m y^n}{R^2 - y^a y^a} \right) dy^m dy^n. \quad (10)$$

To find an easy way to connect the spherical to the Cartesian coordinates, that is the business of the $\text{Cliff}(0, \dim-1)$. The complex case $n = 1$ is well known, so starting with S^3 ,

$$\begin{aligned} q &= x_0 + x_1 E_1 + x_2 E_2 + x_3 E_3 \\ &= |R| \exp(E_1 \theta_1 + E_2 \theta_2 + E_3 \theta_3). \end{aligned} \quad (11)$$

What did we gain? It should be clear that with this special choice of x_μ we can introduce simply the following spherical metric

$$ds^2 = dq d\bar{q} = -R^2 E_i E_j d\theta_i d\theta_j, \quad (12)$$

$$= R^2 \delta_{ij} d\theta_i d\theta_j. \quad (13)$$

For octonions, everything can be done without modifications

$$O = x_0 + x_1 E_1 + \dots + x_7 E_7 \quad (14)$$

$$= |R| \exp(E_1 \theta_1 + \dots + E_7 \theta_7), \quad (15)$$

$$(16)$$

Leading to

$$ds^2 = dO d\bar{O} = -R^2 e_i e_j d\theta_i d\theta_j, \quad (17)$$

$$= R^2 \delta_{ij} d\theta_i d\theta_j. \quad (18)$$

Going to higher dimensions, it is impossible to repeat such constructions. According to **the standard** classification of Clifford Algebras[26], $\text{Cliff}(0, 15)$ is represented by 128×128 matrices.

3 The Self Duality

To start, the 4 dimensional Yang-Mills self duality is

$$F = \pm^* F, \quad (19)$$

or in terms of components

$$F_{\alpha\beta} = \pm \frac{1}{2} \epsilon_{\alpha\beta\mu\nu} F_{\mu\nu}. \quad (20)$$

where $\epsilon_{\alpha\beta\mu\nu}$ is the Levi-Civita tensor. The possible generalization of this self duality is proposed to be

$$F_{\alpha\beta} = \frac{1}{2} \eta_{\alpha\beta\mu\nu} F_{\mu\nu}, \quad (21)$$

where $\eta_{\alpha\beta\mu\nu}$ is a fourth rank tensor. Generally (21) is not $so(dim)$ invariant for $dim > 4$ since a generic $\eta_{\alpha\beta\mu\nu}$ invariant under the action of $so(dim)$ in arbitrary dimensions is not available but it can be invariant with respect to a submanifold of $so(dim)$. Here, we would like to fix this submanifold as S^7 and present a simple method for determining η explicitly.

For octonions, in contrast to quaternions, **our left and right actions do not commute which makes S^7 a non-group manifold** contrary to S^3 . Simply because S^7 possess a varying torsion whereas S^3 has a constant torsion $T(X,Y)$ equal to $-2f_{ijk}$ since for AP spaces

$$\nabla_X Y = 0 \quad (22)$$

leads to

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] = -[X, Y]. \quad (23)$$

In [27, 28, 29, 30, 31], using geometric tools, the notion of soft Lie algebra has been introduced where we no longer have structure constants but structure functions defined by our vector fields and they vary over S^7 . At the level of the algebra such a varying torsion can be seen as follows²:

$$[E_i, E_j] = 2f_{ijk}E_k - 2[E_i, 1|E_j], \quad (24)$$

$$= 2(f_{ijk}1_{8 \times 8} + [E_i, 1|E_j]E_k)E_k, \quad (25)$$

$$= 2\rho_{ijk}E_k. \quad (26)$$

So our structure constants become matrices³ and our torsion is

$$T(X, Y) = -2\rho_{ijk}. \quad (27)$$

Now to build a solitonic model, the idea is simply to gauge the S^7 and to compactify our Euclidean space from \mathcal{R}^8 to S^8 which is a non-trivial manifold and we should cover it by at least two patches (e.g north and south semi-hyper-spheres) then we have to introduce two different gauge fields which are equivalent up to a gauge transformation in the overlapping region (the equator is S^7), so the gauge transformations define a map from the equator to the gauge algebra :

$$S^7 \longrightarrow S^7, \quad (28)$$

we have[34],

$$\pi_{S^7}(S^7) = Z, \quad (29)$$

establishing that our solutions will represent distinct cohomology classes which is the required guarantee for the stability of our instanton i.e our model is not just exactly solvable - integrable - but of solitonic type. In the next section, we are going to prove that our solution satisfies (29).

The critical point for the self-duality condition is the natural existence of the third rank antisymmetric tensor ρ which is important to determine the fourth rank tensor of (21). Adding a zero index to extend ρ_{ijk} from R^7 to R^8 , we define

$$\eta_{0ijk} = \rho_{ijk}, \quad (30)$$

²Eqn. (24) is similar to the starting point for the Knizhnik-Bershadsky arbitrary N superconformal algebra in two dimensions[32, 33]

³There is no summation in the second term of (25) to avoid an ugly 1/7 factor.

and zero elsewhere which enables us to introduce the Clifford self duality condition for a 2-form as

$$F = \pm \star F \iff F_{0\mu} = \pm \frac{1}{2} \eta_{0\mu\alpha\beta} F_{\alpha\beta}, \quad (31)$$

$$\iff F_{0i} = \pm \frac{1}{2} \eta_{0ijk} F_{jk}, \quad (32)$$

$$\iff F_{0i} = \pm \frac{1}{2} \rho_{ijk} F_{jk}. \quad (33)$$

This choice mimicks the 4-dimensional electromagnetic duality

$$F_{0i} = -\epsilon_{ijk} F_{jk}, \quad (34)$$

in contrast to the condition used in the previous octonionic formulations where $\eta_{abcd} = \epsilon_{abcdijk} f_{ijk}$ which is very specific to octonions. Also, it is evident that our condition in 4 dimensions reduces to the standard one.

With the notation (30), E_i is given by

$$(E_i)_{\alpha\beta} = \delta_{i\alpha} \delta_{\beta 0} - \delta_{i\beta} \delta_{\alpha 0} + \eta_{0i\alpha\beta}, \quad (35)$$

in complete agreement with 't Hooft notation's [35] leading to

$$[E_i, E_j] = 2\eta_{0ijk} E_k - 2[E_i, 1|E_j]. \quad (36)$$

We have preferred to work explicitly with octonions but the same construction holds equally well for any Lie group $\geq so(7)$. It is just an embedding problem which can be solved using the following facts :

$$so(7) \sim \left\{ \frac{1}{4} [E_i, E_j] \right\}, \quad so(8) \sim \left\{ \frac{1}{4} [E_i, E_j], \frac{1}{2} E_k \right\}, \quad (37)$$

and

$$\pi_7(so(7)) = Z, \quad \pi_7(so(8)) = Z \oplus Z. \quad (38)$$

4 The 't Hooft Solution

An ansatz for the 't Hooft [35] like solution can be done but let's first construct a self-dual basis. Anyone who has the least knowledge about instantons knows quite well the utility of such tensors and how they may be used as a shortcut for many calculations.

Using octonions, we define

$$E_\mu \equiv (1, E_i) \quad , \quad \bar{E}_\mu \equiv (1, -E_i), \quad (39)$$

and

$$\vartheta_{\mu\nu} = \frac{1}{2} (\bar{E}_\mu E_\nu - \bar{E}_\nu E_\mu). \quad (40)$$

which is Clifford self Dual

$$\star \vartheta = \vartheta, \quad (41)$$

i.e

$$\star \vartheta_{\alpha\beta} = \frac{1}{2} \eta_{\alpha\beta\mu\nu} \vartheta_{\mu\nu} = \vartheta_{\mu\nu}. \quad (42)$$

We can now construct our solution. Consider an S^7 element

$$g(x) = \frac{E_\mu x^\mu}{x^2}, \quad (43)$$

mimicking the quaternionic case, our self-dual gauge is

$$A_\mu(x) = \frac{x^2}{x^2 + \lambda^2} g^{-1}(x) \partial_\mu g(x) = -\frac{\vartheta_{\mu\nu} x^\nu}{\lambda^2 + x^2} \quad (44)$$

leading to

$$F_{\mu\nu} = \frac{\vartheta_{\mu\nu} 2\lambda^2}{(\lambda^2 + x^2)^2}, \quad (45)$$

Which is Clifford self-dual

$$F_{\mu\nu} = \frac{1}{2} \eta_{\mu\nu\alpha\beta} F_{\alpha\beta}. \quad (46)$$

An anti-self-dual can be done easily using an anti-self-dual basis.

Now, the important point, we have to compute our “torsionfull cohomology group”. We know that for a quaternionic instanton it is simply $\pi_3(S^3)$ given by (up to a normalization constant)

$$\int_{S^3} I_1(\alpha, \beta, \gamma) = \int_{S^3} \text{Tr}(\epsilon_{\alpha\beta\gamma} g^{-1} \partial_\alpha g g^{-1} \partial_\beta g g^{-1} \partial_\gamma g) \quad (47)$$

by explicit calculation of any one of these elements, yields

$$\begin{aligned} I_1(1, 2, 3) &= I_1(1, 3, 2) = I_1(2, 1, 3) = I_1(2, 3, 1) = I_1(3, 1, 2) = I_1(3, 2, 1) \\ &= -\frac{4x_0}{x^4}. \end{aligned} \quad (48)$$

For octonions

$$\int_{S^7} I_2(\alpha, \beta, \gamma) = \int_{S^7} \text{Tr}(\rho_{\alpha\beta\gamma} g^{-1} \partial_\alpha g g^{-1} \partial_\beta g g^{-1} \partial_\gamma g) \quad (49)$$

again any of these elements is

$$\begin{aligned} I_2(1, 2, 3) &= I_2(1, 4, 5) = I_2(2, 4, 6) = I_2(3, 4, 7) = \dots = \text{all the possible symmetrization} \\ &= -\frac{8x_0}{x^4}. \end{aligned} \quad (50)$$

It is clear that - apart from a normalization constant - our I_2 is an element of $\pi_7(S^7)$.

Generally the 't Hooft like solution is

$$A_\mu = -\frac{\vartheta_{\mu\nu}(x-y)^\nu}{\lambda^2 + (x-y)^2} \quad (51)$$

where y are eight free parameters because of translation invariance so our solution has $9k$ free parameters.

Having the topological stability criteria, our n -solitons can never decay either to the trivial or any other m -solitons. A generic $(9k + 8)$ n -solitons solution can also be written without problems but a “preliminary” calculation of our moduli space, using the methods developed in this paper and some other Clifford Bundle techniques, indicates that the dimension of the moduli space is $16k - 7$ so a twistor construction similar to the ADHM solution[36] is needed ⁴. We will return to this point elsewhere.

Now, problems start. In 4 dimensions, Instantons have a very clear important meaning. They satisfy the Yang-Mills equation of motion and represent the absolute minima of our non-abelian gauge fields. Working with quaternions E_μ ($\mu = 1 \dots 4$, (the $\times 4$ quaternionic matrices can be found in [21]),

$$A_\mu = \frac{-\partial_{\mu\nu} x^\nu}{\lambda^2 + x^2} \quad (52)$$

which satisfy

$$D_\mu F_{\mu\nu} = \partial_\mu F_{\mu\nu} + [A_\mu, F_{\mu\nu}] = 0. \quad (53)$$

Going to octonions ($\mu = 1 \dots 8$), we find that our solution (44) does not satisfy the Y.M. equation of motion.

$$D_\mu F_{\mu\nu} = \partial_\mu F_{\mu\nu} + [A_\mu, F_{\mu\nu}] \neq 0. \quad (54)$$

It comes too close

$$3\partial_\mu F_{\mu\nu} + [A_\mu, F_{\mu\nu}] = 0, \quad (55)$$

but in physics there is no difference between 3 or 10^{10} . Our solution fail to mimick exactly the 4 dimensional case. In fact from the start, and contrary to the quaternionic case, our Clifford self duality does not imply the Bianchi identities.

But only for $\lambda \rightarrow 0$, the 't Hooft like solution in 8 dimensions satisfy both of the Clifford self duality and the YM eqn. of motion. To this point, omitting such constraint, one should change the form of the solution, for example, a pure gauge connection

$$\begin{aligned} A_\mu = g^{-1} \partial_\mu g &\implies F_{\mu\nu} = \star F_{\mu\nu}, \\ &\implies D_\mu F_{\mu\nu} = 0. \end{aligned} \quad (56)$$

But, at least for the time being, we don't know yet the possibility of such solution. Recently, the authors of [11] complained about the self duality of the Fubini-Nicolai solution [4]. So, the 8 dimensional case is still open. In summary, we solved a new 8 dimensions self duality, that is reducible to 4 dimensional case, but our solution, only in certain limit, satisfy the Yang-Mills equation. Of course the four dimensional case is more powerful because the self duality is related directly to the Bianchi identity which does not hold in higher dimensions.

⁴Actually, this $16k - 7$ can be counted directly from the number of free parameters involved in a twistor like methods

5 Higher Dimensions

Going to higher dimensions, we define a hexagonions (\mathcal{X}) as

$$\mathcal{X} = \mathcal{O}_1 + \mathcal{O}_2 e_8 \quad (57)$$

$$= x_0 e_0 + \dots + x_{16} e_{16}. \quad (58)$$

and

$$e_i e_j = -\delta_{ij} + f_{ijk} e_k. \quad (59)$$

Now, we have to find a suitable form of f_{ijk} . Recalling how this structure constant is written for octonions

$$O = \mathcal{Q}_1 + \mathcal{Q}_2 e_4 \quad (60)$$

$$= x_0 e_0 + \dots + x_7 e_7, \quad (61)$$

from \mathcal{Q} , we have already chosen $e_1 e_2 = e_3$ and from the decomposition of (60), we set $e_1 e_4 = e_5$, $e_2 e_4 = e_6$ and $e_3 e_4 = e_7$, but we are still lacking the relation between the remaining possible triplets, $\{e_1, e_6, e_7\}$; $\{e_2, e_5, e_7\}$; $\{e_3, e_5, e_6\}$ which can be fixed using

$$\begin{aligned} e_1 e_6 &= e_1(e_2 e_4) = -(e_1 e_2) e_4 = -e_3 e_4 = -e_7, \\ e_2 e_5 &= e_2(e_1 e_4) = -(e_2 e_1) e_4 = +e_3 e_4 = +e_7, \\ e_3 e_5 &= e_3(e_1 e_4) = -(e_3 e_1) e_4 = -e_2 e_4 = -e_6. \end{aligned} \quad (62)$$

That is all for octonions. Going up to \mathcal{X} , we have the seven octonionic conditions, and the decomposition (57) gives us $e_1 e_8 = e_9$, $e_2 e_8 = e_A$, $e_3 e_8 = e_B$, $e_4 e_8 = e_C$, $e_5 e_8 = e_D$, $e_6 e_8 = e_E$, $e_7 e_8 = e_F$ where $A = 10$, $B = 11$, $C = 12$, $D = 13$, $E = 14$ and $F = 15$. The other elements of the multiplication table may be chosen in analogy with (62), explicitly, the 35 Hexagonionic triplets N are

$$\begin{aligned} &(123), (145), (246), (347), (257), (176), (365), \\ &(189), (28A), (38B), (48C), (58D), (68E), (78F), \\ &(1BA), (1DC), (1EF), (29B), (2EC), (2FD), (3A9), \\ &(49D), (4AE), (4BF), (3FC), (3DE), (5C9), (5AF), \\ &(5EB), (6FD), (6CA), (6BD), (79E), (7DA), (7CB), \end{aligned} \quad (63)$$

and so on for any generic higher dimensional “field” \mathcal{F}^n .

In general, from some combinatorics, the number of such triplets for a general \mathcal{F}^n field is ($n > 1$)

$$N = \frac{(2^n - 1)!}{(2^n - 3)! \cdot 3!}, \quad (64)$$

giving

\mathcal{F}^n	n	dim	N
\mathcal{Q}	2	4	1
\mathcal{O}	3	8	7
\mathcal{X}	4	16	35

and so on.

(65)

One may notice that for any non-ring division algebra $(\mathcal{F}, n > 3)$, $N > \dim(\mathcal{F}^n)$ except when $\dim = \infty$ i.e a functional Hilbert space with a $\text{Cliff}(0, \infty)$ structure. Does this inequality have any relation with the ring division structure of the (S^1, S^3, S^7) spheres ?! Yes, that is what we are going to show now : Following, the same translation⁵ idea - projecting our algebra over $R^{16}[21]$ - any E_i is given by similar relation as for \mathcal{Q} or \mathcal{O}

$$(E_i)_{\alpha\beta} = \delta_{i\alpha}\delta_{\beta 0} - \delta_{i\beta}\delta_{\alpha 0} + f_{i\alpha\beta}. \quad (66)$$

But contrary to the quaternions and octonions, the Clifford algebra closes for a subset of these E_i 's, namely

$$\{E_i, E_j\} = -2\delta_{ij} \quad \text{for } i, j, k = 1 \dots 8. \quad (67)$$

Because, we have lost the ring division structure. Also, notice that (67) is in agreement with the Clifford algebra classification[26]. Following this method, we can give a simple way to write real Clifford algebras over any arbitrary dimensions.

Sometimes, a specific multiplication table may be favoured. As we are interested in solitons, the existence of a symplectic structure - related to the bihamiltonian formulation of integrable models - should be welcome. It is known from the Darboux theorem, that locally a symplectic structure is given up to a minus sign by

$$\mathcal{J}_{\dim \times \dim} = \begin{pmatrix} 0 & -1_{\frac{\dim}{2} \times \frac{\dim}{2}} \\ 1_{\frac{\dim}{2} \times \frac{\dim}{2}} & 0 \end{pmatrix}, \quad (68)$$

that fixes the following structure constants

$$f_{(\frac{\dim}{2})1(\frac{\dim}{2}+1)} = -1, \quad (69)$$

$$f_{(\frac{\dim}{2})2(\frac{\dim}{2}+2)} = -1, \quad (70)$$

$$\dots \quad (71)$$

$$f_{(\frac{\dim}{2})(\frac{\dim}{2}-1)(\dim-1)} = -1, \quad (72)$$

which is clearly the decomposition that we have chosen in (60) for octonions

$$f_{415} = f_{426} = f_{437} = -1. \quad (73)$$

Generally our symplectic structure is

$$\left(1|E_{(\frac{\dim}{2})}\right)_{\alpha\beta} = \delta_{0\alpha}\delta_{\beta(\frac{\dim}{2})} - \delta_{0\beta}\delta_{\alpha(\frac{\dim}{2})} - \epsilon_{\alpha\beta(\frac{\dim}{2})}. \quad (74)$$

Moreover some other choices may exhibit a relation with number theory and Galois fields [38]. It is highly non-trivial how Clifford algebraic language can be used to unify many distinct mathematical notions such as Grassmanian [22], complex, quaternionic and symplectic structures.

⁵ The translation idea was given first in [37] from \mathcal{Q} to \mathcal{C}^2 in the context of quaternionic quantum mechanics.

6 Conclusion

Octonions have a central in mathematics and they play a vital role for $D = 11$ supergravity compactification. Understanding their real job in physics is highly needed, espically, with the recent string dualities. In this article, the important message is to use the associative non Lie algebra $\text{Cliff}(0,7)$ instead of octonions which led us to structure matrices. Once this is accepted. The road is open. For example, using this Clifford language, we have

$$E_a E_b = \epsilon_{abcdefg} E_c E_d E_e E_f E_g. \quad (75)$$

extending it to R^8 , we have a natural 8 dimensions Levi-Civita and we may study a self duality relation for the Reimannian tensor

$$R_{\alpha\beta\gamma\delta} = \epsilon_{\alpha\beta\gamma\delta\zeta\eta\mu\nu} R_{\zeta\eta\mu\nu}. \quad (76)$$

So, in 2 dimensions, we have a dual σ model (constrained scalar field), in 4 dimensions, a dual YM field (constrained spin one field) and lastly in 8 dimensions, a dual gravitational field (constrained spin 2 field) !

Once we write the self duality relation in a certain higher dimension, then we can recover the lower cases by trivial dimension reduction. A very interesting situation happens in odd dimensions, instead of considering the self duality condition (31), we may generalize the Chern-Simons form to 7 dimensions as

$$Ch = \rho_{ijk}(A_i A_j A_k - g A_i \partial_j A_k), \quad (77)$$

for suitable g to be defined appropriately in correspondence with λ . The first part of (77) is the natural generalization of the WZNW term to octonions⁶ whereas the second part is the octonionic Hopf term. Also, we may define a vector product and a curl operator over R^7 or any of its subspaces by

$$A_i = \rho_{ijk} A_j A_k \quad ; \quad B_i = \rho_{ijk} \partial_j A_k. \quad (78)$$

The 7 dimensional monopole is just the Clifford self-duality, making the $A_{1\dots 7}$ static and defining $A_0 = \phi$ our Higgs field.

We mentioned the $so(n)$ series embedding but its extension to any Lie algebra should be clear as

$$su(n) \subset so(2n), \quad (79)$$

$$sp(n) \subset so(4n), \quad (80)$$

and having the following interesting topological facts (taking into account Bott periodicity)

$$\pi_{2^n-1}(so(2^n-1)) = Z, \quad (81)$$

$$\pi_{2^n-1}(so(2^n)) = Z \oplus Z, \quad (82)$$

$$\pi_{2^n-1}(su(2^n)) = Z, \quad (83)$$

$$\pi_{2^n-1}(sp(2^{n-1})) = Z, \quad (84)$$

⁶Note that the natural dimension of an octonionic sigma model is 6 not 2 dimensions as $\pi_3(S^7) \neq Z$ which may be related to 5-brane.

also, noticing that

$$\pi_{15}(H) = \mathbb{Z} \quad \text{for} \quad H = F_4, E_6, E_7, E_8, \quad (85)$$

which hopefully may be related to the $E_8 \times E_8$ string solitons [8, 9, 10, 7, 11].

As physics speaks mathematics, we tried our best to adopt Dirac's point of view: "one can generalize his physics by generalizing his mathematics", a line of attack that always proved to be useful.

Just before the submission of this work, we received a preprint[39] where a $\text{Cliff}(0,7)$ is also used instead of octonions. But it is clear that we addressed different questions related to the 8 dimensional instanton.

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